NAKANO POSITIVITY AND THE L^2-METRIC ON THE DIRECT IMAGE OF AN ADJOINT POSITIVE LINE BUNDLE

INDRANIL BISWAS AND CHRISTOPHE MOUROUGANE

1. Introduction

A holomorphic vector bundle E over a complex projective manifold X is called *ample* if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ over the projective bundle $\mathbb{P}(E)$ of hyperplanes in E is ample. This notion of ampleness was introduced by R. Hartshorne in [Ha].

On the other hand, P.A. Griffiths in [Gr] introduced an analytic notion of positivity of a vector bundle. A holomorphic vector bundle E over X is called *Griffiths positive* if it admits a Hermitian metric h such that the curvature $C_h(E) \in C^\infty(X, \Omega^{1,1}_X \bigotimes \mathrm{End}(E))$ of the Chern connection on E for the Hermitian structure h has the property that for every $x \in X$ and every nonzero holomorphic tangent vector $0 \neq v \in T_xX$, the endomorphism $\sqrt{-1} \cdot C_h(E)(x)(v, \bar{v})$ of the fiber E_x is positive definite with respect to h.

If h is a Griffiths positive Hermitian metric on E, then the Hermitian metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ induced by h has positive curvature. Therefore, E is ample by a theorem due to Kodaira. An ample line bundle admits a Griffiths positive Hermitian metric. Also, an ample vector bundle on a Riemann surface is Griffiths positive, which was proved by H. Umemura in [Um]. However, the question posed by Griffiths [Gr, page 186, (0.9)] asking whether every ample vector bundle is Griffiths positive is yet to be settled.

The notion of Griffiths positivity was strengthened by S. Nakano. A holomorphic Hermitian vector bundle (E, h) is called *Nakano positive* if $\sqrt{-1} \cdot C_h(E)$ is a positive form on $TX \bigotimes E$. A Nakano positive vector bundle is clearly Griffiths positive. In the other direction, H. Skoda and J.-P. Demailly proved that if E is Griffiths positive, then $E \otimes \det E$ is Nakano positive [DS], $\det E$ being the line bundle given by the top exterior power of E.

Our aim here is to establish the following analytic property of an ample vector bundle (Theorem 3.1).

Theorem A. *Let E be an ample vector bundle of rank r over a projective manifold X. The vector bundle $S^k(E) \bigotimes \det E$ is Nakano positive for every $k \geq 0$, where $S^k(E)$ denotes the k-th symmetric power of E. More generally, for every decreasing sequence $\lambda \in \mathbb{N}^r$ of height (= number of positive elements) l, the vector bundle $\Gamma^\lambda E \bigotimes (\det E)^{\otimes l}$*

is Nakano positive; $\Gamma^\lambda E$ *is the vector bundle corresponding to the irreducible representation of* $GL(r,\mathbb{C})$ *defined by the weight* λ*. So, in particular,* $\Gamma^\lambda E \bigotimes (\det E)^{\otimes l}$ *is Griffiths positive.*

In the special case where X is a toric variety or an abelian variety, the Griffiths positivity of such vector bundles associated to an ample bundle was proved in [Mo].

The above Theorem A is obtained as consequence of a result on the Nakano positivity of the direct image of an adjoint positive line bundle. This result on Nakano positivity of direct image will be described next.

Let L be a holomorphic line bundle, equipped with a Hermitian metric h, over a connected projective manifold M. Given any section $t \in H^0(M,\, L \bigotimes K_M)$, where K_M is the canonical bundle, its conjugate $\overline{t}$ is realized as a section of $L^* \bigotimes \overline{K_M}$ using h. Now, given another section $s \in H^0(M,\, L \bigotimes K_M)$, consider the top form on M obtained from $s \wedge \overline{t}$ by contracting L with L^*. The L^2 inner product on $H^0(M,\, L \bigotimes K_M)$ is defined by taking the integral of this form over M.

We prove the following theorem on the Nakano positivity of the L^2 metric on a direct image (Theorem 2.3).

Theorem B. *Let* $\psi : Y \longrightarrow X$ *be a holomorphically locally trivial fiber bundle, where* X *and* Y *are connected projective manifolds, and* $H^1(\psi^{-1}(x),\, \mathbb{Q}) = 0$ *for some, hence every, point* $x \in X$*. Let* L *be a holomorphic line bundle over* Y *equipped with a positive Hermitian metric* h*. Then the* L^2*-metric, defined using* h*, on the vector bundle* $\psi_*(K_{Y/X} \bigotimes L)$ *over* X *is Nakano positive;* $K_{Y/X}$ *is the relative canonical bundle.*

We note that the condition of ampleness of L in Theorem B ensures that the direct image $\psi_*(K_{Y/X} \bigotimes L)$ is locally free with the fiber of the corresponding vector bundle over any point $x \in X$ being $H^0(\psi^{-1}(x),\, (K_{Y/X} \bigotimes L)\big|_{\psi^{-1}(x)})$.

Theorem A is an immediate consequence of Theorem B applied to a natural line bundle over the flag bundle $\psi : M_\lambda(E) \longrightarrow X$ associated to an ample vector bundle E by a weight λ. In particular, setting $Y = \mathbb{P}(E)$ and $L = \mathcal{O}_{\mathbb{P}(E)}(k+r)$ in Theorem B, where E is an ample vector bundle of rank r, the Nakano positivity of $S^k(E) \bigotimes \det E$ is obtained. This particular case corresponds to the weight $(k, 0, 0, \cdots, 0)$.

Acknowledgments: Remarks 3.3 and 3.5 were communicated to us by Jean-Pierre Demailly. We are very grateful to him for this. We thank M.S. Narasimhan for some useful discussions. We would like to thank the International Centre for Theoretical Physics, Trieste, for its hospitality. The first-named author also thanks Joseph Le Potier and the Institut de Mathématiques de Jussieu for hospitality. The second-named author thanks T. Ohsawa and K. Yoshikawa for his visit to Nagoya University.

2. Curvature of the L^2-metric

We first recall the definition of Nakano positivity. Let E be a holomorphic vector bundle over X equipped with a Hermitian metric h. Let $C_h(E)$ denote the curvature of the corresponding Chern connection on E. Let Θ_h denote the unique sesquilinear form on $TX \bigotimes E$ such that for any $v_1, v_2 \in T_xX$ and $e_1, e_2 \in E_x$, the equality

$$\Theta_h(v_1 \otimes e_1, v_2 \otimes e_2) \;=\; \sqrt{-1} \cdot \langle C_h(E)(v_1, \overline{v_2})e_1 \,, e_2\rangle_h$$

is valid. The metric h is called *Nakano positive* if the sesquilinear form Θ_h on $TX \bigotimes E$ is Hermitian, i.e., it is positive definite.

Let X and Y be two connected complex projective manifolds of dimension m and n respectively, and let

$$\psi \,:\, Y \longrightarrow X \tag{2.1}$$

be a holomorphic submersion defining a holomorphically locally trivial fiber bundle over X of relative dimension $f = n - m$. So, every point of X has an analytic open neighborhood U such that $\psi^{-1}(U)$ is holomorphically isomorphic to the trivial fiber bundle $U \times F$ over U, where F is the typical fiber of ψ. The relative canonical line bundle $K_Y \bigotimes \psi^* K_X^{-1}$ over Y will be denoted by $K_{Y/X}$.

For any $x \in X$, the submanifold $\psi^{-1}(x)$ of Y will be denoted by Y_x.

Let L be an ample line bundle over Y. The direct image $\psi_*(K_{Y/X} \bigotimes L)$ is locally free on X. Indeed, from the Kodaira vanishing theorem it follows that all the higher direct images of $K_{Y/X} \bigotimes L$ vanish. Let V denote the vector bundle over X given by this direct image $\psi_*(K_{Y/X} \bigotimes L)$.

Fix a positive Hermitian metric h on L. For any point $x \in X$, using the natural conjugate linear isomorphism of $\Omega^{f,0}_{Y_x}$ with $\Omega^{0,f}_{Y_x}$ and the Hermitian metric h on $L|_{Y_x}$, a conjugate linear isomorphism between $\Omega^{f,0}_{Y_x} \bigotimes L|_{Y_x}$ and $\Omega^{0,f}_{Y_x} \bigotimes L^*|_{Y_x}$ is obtained. We denote this conjugate linear isomorphism by ι.

The L^2 Hermitian metric on the vector bundle $V := \psi_*(K_{Y/X} \bigotimes L)$ is defined by sending any pair of sections

$$t_1,\, t_2 \,\in\, V_x \,:=\, H^0(Y_x\,,\, (K_{Y/X} \otimes L)\big|_{Y_x})$$

to the integral over Y_x of the contraction of $(\sqrt{-1})^{f^2} t_1$ with $\iota(t_2)$. In other words, if $t_i = s_i \otimes d\zeta_1 \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_f$, where $\{\zeta_1, \cdots, \zeta_f\}$ is a local holomorphic coordinate chart on the fiber Y_x and s_i, $i = 1, 2$, is a local section of L, then the pairing

$$\langle t_1, t_2\rangle \,:=\, (\sqrt{-1})^{f^2} \int_{Y_x} \langle s_1, s_2\rangle_h \cdot d\zeta_1 \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_f \wedge d\overline{\zeta_1} \wedge d\overline{\zeta_2} \wedge \cdots \wedge d\overline{\zeta_f} \tag{2.2}$$

is the L^2 inner product of the two vectors t_1 and t_2 of the fiber V_x. The top form

$$\langle s_1, s_2\rangle_h \cdot d\zeta_1 \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_f \wedge d\overline{\zeta_1} \wedge d\overline{\zeta_2} \wedge \cdots \wedge d\overline{\zeta_f}$$

clearly depends only on t_1 and t_2 and, in particular, it does not depend on the choice of the coordinate function ζ; in other words, it is a globally defined top form on Y_x. We note that the definition of L^2-metric does not require a metric on Y_x.

Our aim is to compute the curvature of the Chern connection on V for the L^2 metric.

Theorem 2.3. *Assume that $H^1(Y_x, \mathbb{Q}) = 0$ for some, hence every, point $x \in X$. Then the L^2-metric on the direct image V is Nakano positive.*

Proof. Take a point $\xi \in X$. Let $\{z_1, z_2, \cdots, z_m\}$ be a holomorphic coordinate chart on X around ξ such that $\xi = 0$.

Let r be the rank of the direct image V. Fix a normal frame $\{t_1, t_2, \cdots, t_r\}$ of V around ξ with respect to the L^2 metric. In other words, $\{t_1, t_2, \cdots, t_r\}$ is a holomorphic frame of V around ξ such that for the function $\langle t_\alpha, t_\beta\rangle$ around ξ we have

$$\begin{aligned} \langle t_\alpha, t_\beta\rangle\Big|_{z=0} &= \delta_{\alpha\beta} \\ \frac{\partial\langle t_\alpha, t_\beta\rangle}{\partial z_i}\Big|_{z=0} &= 0 \end{aligned}$$

for all $\alpha, \beta \in [1, r]$ and all $i \in [1, m]$. We note that the second condition is equivalent to the condition that $d\langle t_\alpha, t_\beta\rangle(0) = 0$, where d is the exterior derivation.

Let ∇^{L^2} denote the Chern connection on V for the L^2 metric. Its curvature, which is a $\mathrm{End}(V)$-valued $(1,1)$-form on X, will be denoted by C_{L^2}. Take any vector

$$v = \sum_{i=1}^{m}\sum_{\alpha=1}^{r} v_{i,\alpha}\frac{\partial}{\partial z_i} \otimes t_\alpha \in T_\xi X \otimes V_\xi .$$

We wish to show that

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{\alpha=1}^{r}\sum_{\beta=1}^{r} v_{i,\alpha}\overline{v_{j,\beta}}\sqrt{-1}\cdot\left\langle C_{L^2}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z_j}}\right) t_\alpha\, , t_\beta\right\rangle_{L^2} > 0 \tag{2.4}$$

with the assumption that $v \neq 0$. Now, we have

$$\left\langle \sqrt{-1}\cdot C_{L^2}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z_j}}\right) t_\alpha, t_\beta\right\rangle_{L^2}(0) = \frac{1}{\sqrt{-1}}\cdot\frac{\partial^2\langle t_\alpha\, , t_\beta\rangle}{\partial z_i\partial\overline{z_j}}(0)\,.$$

For any $\alpha \in [1, r]$, let $\widehat{t_\alpha}$ be the *unique* section of $K_{Y/X} \bigotimes L$, defined on an analytic open neighborhood of $Y_\xi := \psi^{-1}(\xi)$, which is determined by the condition that for any $x \in X$ in a sufficiently small neighborhood of ξ, the restriction of $\widehat{t_\alpha}$ to the fiber $\psi^{-1}(x)$ represents $t_\alpha(x)$, the evaluation of the section t_α at x. The holomorphicity of the section t_α of V implies that the section $\widehat{t_\alpha}$ of $K_{Y/X} \bigotimes L$ is also holomorphic.

Fix a trivialization of the fiber bundle over a neighborhood of ξ defined by the projection ψ. In other words, we fix an isomorphism of the fiber bundle $\psi^{-1}(U) \longrightarrow U$ over some open subset $U \subset X$ containing ξ with the trivial fiber bundle $U \times Y_\xi$ over U. Using this isomorphism, the relative tangent bundle for the projection ψ gets identified

with a subbundle of the restriction of the tangent bundle TY to $\psi^{-1}(U)$. Consequently, any relative differential form on $\psi^{-1}(U)$ becomes a differential form on $\psi^{-1}(U)$.

If θ (respectively, ω) is a L-valued (i_1, i_2)-form (respectively, L-valued (j_1, j_2)-form), i.e., a smooth section of $\Omega_Y^{i_1,i_2} \bigotimes L$ (respectively, $\Omega_Y^{j_1,j_2} \bigotimes L$), then define $\{\theta, \omega\}$ to be the $(i_1 + j_2, i_2 + j_1)$-form on Y obtained from the section

$$\theta \wedge \overline{\omega} \in C^\infty(\Omega_Y^{i_1+j_2,i_2+j_1} \otimes L \otimes L^*)\,,$$

where the conjugate of L has been identified with L^* using the Hermitian structure h, and then contracting L with L^*.

In the above notation we have

$$\partial_X\langle t_\alpha, t_\beta\rangle = \psi_*(\partial_Y\{\widehat{t}_\alpha, \widehat{t}_\beta\})\,, \tag{2.5}$$

where ψ_* is the integration of forms along the fiber (the Gysin map) and $\alpha, \beta \in [1, r]$.

It is easy to see directly that the right-hand side of (2.5) does not depend on the choice of the trivialization of the fibration defined by ψ. Indeed, any two choices of the local trivialization defines a homomorphism, over U, from the tangent bundle TU to the trivial vector bundle over U with the space of vertical vector fields $H^0(Y_\xi,\, TY_\xi)$ as the fiber. This homomorphism is constructed by taking the difference of the two horizontal lifts, given by the two trivializations, of vector fields. If the one-form in the right-hand side of (2.5) for a different choice of trivialization is denoted by η, then the one-form defined by the difference

$$\eta - \psi_*(\partial_Y\{\widehat{t}_\alpha, \widehat{t}_\beta\})$$

sends any tangent vector $v \in T_xX$ to

$$\int_{\psi^{-1}(x)} L_{\hat{v}}\{\widehat{t}_\alpha, \widehat{t}_\beta\}\,,$$

where $L_{\hat{v}}$ is the Lie derivative with respect to the vertical vector field corresponding to v for the given pair trivializations of the fiber bundle. Now the identity $L_{\hat{v}} = d\circ i_{\hat{v}} + i_{\hat{v}}\circ d$ and the Stokes' theorem together ensure that the right-hand side of (2.5) is independent of the choice of trivialization of the fibration defined by ψ.

Let ∇ denote the Chern connection of the holomorphic line bundle L equipped with the Hermitian metric h. Its curvature will be denoted by C_h.

Since $\widehat{t}_\alpha$ is a holomorphic section for every $\alpha \in [1, r]$, the equality

$$\psi_*(\partial_Y\{\widehat{t}_\alpha, \widehat{t}_\beta\}) = \psi_*(\{\nabla\widehat{t}_\alpha, \widehat{t}_\beta\})$$

is valid for all $\alpha, \beta \in [1, r]$.

For any $i \in [1, m]$, let $\frac{\widetilde{\partial}}{\partial z_i}$ denote vector field on $\psi^{-1}(U)$ given by the lift of the local vector field $\frac{\partial}{\partial z_i}$ on $U \subset X$ using the chosen trivialization of the fiber bundle. The contraction of a one-form θ with a vector field ν will be denoted by (θ, ν).

The above equality combined with (2.5) gives

$$\frac{\partial}{\partial z_i}\langle t_\alpha, t_\beta\rangle = \left(\psi_*(\{\nabla_{\frac{\widetilde{\partial}}{\partial z_i}}\widehat{t}_\alpha, \widehat{t}_\beta\})\right)\Big|_{Y_\xi}$$

as any $\widehat{t}_\alpha$ is a *relative top form* with values in L. Now taking taking $\overline{\partial}_Y$ of this equality yields

$$\frac{\partial^2}{\partial\overline{z_j}\partial z_i}\langle t_\alpha, t_\beta\rangle = \left(\overline{\partial}_X\psi_*(\{\nabla_{\frac{\widetilde{\partial}}{\partial z_i}}\widehat{t}_\alpha, \widehat{t}_\beta\}), \frac{\partial}{\partial\overline{z_j}}\right) = \left(\psi_*(\overline{\partial}_Y\{\nabla_{\frac{\widetilde{\partial}}{\partial z_i}}\widehat{t}_\alpha, \widehat{t}_\beta\}), \frac{\partial}{\partial\overline{z_j}}\right).$$

Since any $\widehat{t}_\alpha$ is holomorphic, the last term coincides with

$$\left(\psi_*(\{\nabla^{0,1}\nabla_{\frac{\widetilde{\partial}}{\partial z_i}}\widehat{t}_\alpha, \widehat{t}_\beta\}), \frac{\partial}{\partial\overline{z_j}}\right) + (-1)^f\cdot\left(\psi_*(\{\nabla_{\frac{\widetilde{\partial}}{\partial z_i}}\widehat{t}_\alpha, \nabla^{1,0}\widehat{t}_\beta\}), \frac{\partial}{\partial\overline{z_j}}\right)$$

$$= \psi_*(\{\nabla_{\frac{\widetilde{\partial}}{\partial\overline{z_j}}}\nabla_{\frac{\widetilde{\partial}}{\partial z_i}}\widehat{t}_\alpha\,, \widehat{t}_\beta\}) + (-1)^f\cdot\psi_*(\{\nabla_{\frac{\widetilde{\partial}}{\partial z_i}}\widehat{t}_\alpha\,, \nabla_{\frac{\widetilde{\partial}}{\partial z_j}}\widehat{t}_\beta\}), \tag{2.6}$$

where $\frac{\widetilde{\partial}}{\partial\overline{z_j}}$, as before, denotes the lift of $\frac{\partial}{\partial\overline{z_j}}$ using the chosen trivialization of the fibration, and f is the relative dimension. The holomorphicity of any $\widehat{t}_\alpha$ yields the equality

$$\psi_*(\{\nabla_{\frac{\widetilde{\partial}}{\partial\overline{z_j}}}\nabla_{\frac{\widetilde{\partial}}{\partial z_i}}\widehat{t}_\alpha, \widehat{t}_\beta\}) = -\psi_*(C_h(\frac{\widetilde{\partial}}{\partial z_i}, \frac{\widetilde{\partial}}{\partial\overline{z_j}})\{\widehat{t}_\alpha\,, \widehat{t}_\beta\}).$$

Now, for any $v = \sum_{i=1}^m\sum_{\alpha=1}^r v_{i,\alpha}\frac{\partial}{\partial z_i}\otimes t_\alpha \in T_\xi X\otimes V_\xi$ we have

$$\sum_{i=1}^m\sum_{j=1}^m\sum_{\alpha=1}^r\sum_{\beta=1}^r v_{i,\alpha}\overline{v_{j,\beta}}\cdot\psi_*(C_h(\frac{\widetilde{\partial}}{\partial z_i}, \frac{\widetilde{\partial}}{\partial\overline{z_j}})\{\widehat{t}_\alpha, \widehat{t}_\beta\}) = \psi_*(\sum_{i,j=1}^m C_h(\frac{\widetilde{\partial}}{\partial z_i}, \frac{\widetilde{\partial}}{\partial\overline{z_j}})\{\theta_i, \overline{\theta_j}\}),$$

where $\theta_i := \sum_{\alpha=1}^r v_{i,a}\widehat{t}_a$ is the section of $K_{Y/X}\bigotimes L$ defined on a neighborhood of Y_ξ. Let e be a local section of L defined around a point $y\in Y$ and $\{\zeta_1, \zeta_2, \cdots, \zeta_f\}$ be a holomorphic coordinate chart on Y_ξ around y. Set τ to be the local section of TY defined around y that satisfies the equality

$$d\zeta_1\wedge\zeta_2\wedge\cdots\wedge\zeta_f\otimes e\otimes\tau = \sum_{i=1}^m\frac{\widetilde{\partial}}{\partial z_i}\otimes\theta_i$$

of local sections of $K_{Y/X}\bigotimes L\bigotimes TY$. Using this notation, the evaluation at ξ of the function defined on a neighborhood of ξ by the fiber integral

$$\psi_*(\{\nabla_{\frac{\widetilde{\partial}}{\partial\overline{z_j}}}\nabla_{\frac{\widetilde{\partial}}{\partial z_i}}\widehat{t}_\alpha\,, \widehat{t}_\beta\})$$

in (2.6) coincides with

$$\psi_*\left(\Big\langle C_h(\tau,\overline{\tau})e, e\Big\rangle_h\cdot d\zeta_1\wedge d\zeta_2\wedge\cdots\wedge d\zeta_f\wedge d\overline{\zeta_1}\wedge d\overline{\zeta_2}\wedge\cdots\wedge d\overline{\zeta_f}\right). \tag{2.7}$$

The curvature C_h is given to be positive. So using the expression (2.7) we conclude that the evaluation at ξ of the complex valued function around ξ defined by

$$\frac{1}{\sqrt{-1}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{\alpha=1}^{r}\sum_{\beta=1}^{r} v_{i,\alpha}\overline{v_{j,\beta}}\cdot\psi_*(\{\nabla_{\overline{\frac{\widetilde{\partial}}{\partial z_j}}}\nabla_{\frac{\widetilde{\partial}}{\partial z_i}}\widehat{t}_\alpha\,,\widehat{t}_\beta\})$$

is positive real; $\psi_*(\{\nabla_{\overline{\frac{\widetilde{\partial}}{\partial z_j}}}\nabla_{\frac{\widetilde{\partial}}{\partial z_i}}\widehat{t}_\alpha\,,\widehat{t}_\beta\})$ is a term in (2.6). Consequently, to prove the theorem it suffices to show that the last term in (2.6), namely the complex valued function

$$\psi_*(\{\nabla_{\frac{\widetilde{\partial}}{\partial z_i}}\widehat{t}_\alpha\,,\nabla_{\frac{\widetilde{\partial}}{\partial z_j}}\widehat{t}_\beta\})\,, \tag{2.8}$$

which defined around ξ, actually vanishes at ξ.

Let ω denote the Kähler form on Y_ξ obtained from the curvature of the Chern connection C_h on L. The second part of the normal frame condition gives

$$\Big\langle(\nabla_{\frac{\widetilde{\partial}}{\partial z_i}}\widehat{t}_\alpha)\Big|_{Y_\xi},\widehat{t}_\beta\Big|_{Y_\xi}\Big\rangle_\omega \,=\, \psi_*(\{\nabla_{\frac{\widetilde{\partial}}{\partial z_i}}\widehat{t}_\alpha,\widehat{t}_\beta\})(\xi)\,=\,0$$

for all $\alpha,\beta\in[1,r]$ and $i\in[1,m]$; here $\langle -,-\rangle_\omega$ is the inner product defined using ω. We note that the above orthogonality condition does not depend on the choice of the Kähler form on Y_ξ that is needed to define the orthogonality condition.

The above assertion that the smooth section

$$\widetilde{t}_{i,\alpha}\,:=\,(\nabla_{\frac{\widetilde{\partial}}{\partial z_i}}\widehat{t}_\alpha)\Big|_{Y_\xi}$$

is orthogonal to $H^0(Y_\xi,\,(K_{Y/X}\otimes L)\big|_{Y_\xi})$ is equivalent to the condition that $\widetilde{t}_{i,\alpha}$ is orthogonal to the space $H^{f,0}(Y_\xi,\,L\big|_{Y_\xi})$ of Δ''_{Y_ξ}-harmonic $(f,0)$-forms with values in L; here f, as before, is the relative dimension. This Laplacian Δ''_{Y_ξ} corresponds to the Dolbeault operator on $L\big|_{Y_\xi}$ endowed with the Hermitian metric $h\big|_{Y_\xi}$.

The above orthogonality condition implies that the equality

$$\widetilde{t}_{i,\alpha}\,=\,\Delta''_{Y_\xi}G''_{Y_\xi}(\widetilde{t}_{i,\alpha})\,=\,(D''_{Y_\xi})^*D''_{Y_\xi}G''_{Y_\xi}(\widetilde{t}_{i,\alpha})$$

is valid; here G''_{Y_ξ} is the Green operator corresponding to Δ''_{Y_ξ}. Therefore, we have the following equality

$$\psi_*(\{\nabla_{\frac{\widetilde{\partial}}{\partial z_i}}\widehat{t}_\alpha,\nabla_{\frac{\widetilde{\partial}}{\partial z_j}}\widehat{t}_\beta\})=\int_{Y_\xi}\langle D''_{Y_\xi}(\widetilde{t}_{i,\alpha}),D''_{Y_\xi}G''_{Y_\xi}(\widetilde{t}_{j,\beta})\rangle_\omega=\int_{Y_\xi}\langle(D''_{Y_\xi})^*D''_{Y_\xi}(\widetilde{t}_{i,\alpha}),G''_{Y_\xi}(\widetilde{t}_{j,\beta})\rangle_\omega$$

for the function defined by (2.8).

Now, to prove that the function defined in (2.8) vanishes at ξ it suffices to establish the equality

$$D''_{Y_\xi}(\widetilde{t}_{i,\alpha})\,=\,0 \tag{2.9}$$

for all $i\in[1,m]$ and $\alpha\in[1,r]$.

To prove this we first note that

$$D''_{Y_\xi}(\widetilde{t}_{i,\alpha}) = D''_{Y_\xi}((\nabla_{\frac{\widetilde{\partial}}{\partial z_i}}\widehat{t}_\alpha)\Big|_{Y_\xi}) = (\sum_{k=1}^{f}\nabla_{\frac{\partial}{\partial\overline{\zeta_k}}}(\nabla_{\frac{\widetilde{\partial}}{\partial z_i}}\widehat{t}_\alpha)d\overline{\zeta_k})\Big|_{Y_\xi} = -\sum_{k=1}^{f}C_h(\frac{\widetilde{\partial}}{\partial z_i},\frac{\partial}{\partial\overline{\zeta_k}})\widehat{t}_\alpha d\overline{\zeta_k}\,,$$

where $\{\zeta_1,\cdots,\zeta_{f-1},\zeta_f\}$, as before is a local holomorphic coordinate chart on Y_ξ.

The locally defined $(0,1)$-form $\sum_{k=1}^{f}C_h(\frac{\widetilde{\partial}}{\partial z_i},\frac{\partial}{\partial\overline{\zeta_k}})d\overline{\zeta_k}$ is easily seen to be independent of the coordinate function $\{\zeta_1,\cdots,\zeta_f\}$. Indeed, this locally defined $(0,1)$-form coincides with the contraction

$$i_{\frac{\widetilde{\partial}}{\partial z_i}}C_h|_{Y_\xi}$$

of the $(1,1)$-form C_h with the vector field $\frac{\widetilde{\partial}}{\partial z_i}$.

We have $H^{0,1}(Y_\xi)=0$, as, by assumption, $H^1(Y_\xi,\,\mathbb{Q})=0$ and Y_ξ is Kähler. Consequently, there is no nonzero harmonic form of type $(0,1)$ on Y_ξ. Therefore, the equality (2.9) is an immediate consequence of the following lemma.

Lemma 2.10. *The $(0,1)$-form $i_{\frac{\widetilde{\partial}}{\partial z_i}}C_h|_{Y_\xi}$ on Y_ξ is harmonic.*

Proof of Lemma 2.10. This form $i_{\frac{\widetilde{\partial}}{\partial z_i}}C_h|_{Y_\xi}$ is evidently $\overline{\partial}_{Y_\xi}$-closed.

To prove that it is also $\overline{\partial}^*_{Y_\xi}$-closed, first note that the Kähler identity gives

$$\sqrt{-1}\cdot\overline{\partial}^*_{Y_\xi}\left(\sum_{k=1}^{f}C_h(\frac{\widetilde{\partial}}{\partial z_i},\frac{\partial}{\partial\overline{\zeta_k}})d\overline{\zeta_k}\right) = \Lambda_\omega\partial_{Y_\xi}\sum_{k=1}^{f}C_h(\frac{\widetilde{\partial}}{\partial z_i},\frac{\partial}{\partial\overline{\zeta_k}})d\overline{\zeta_k}\,.$$

The right-hand side coincides with

$$\Lambda_\omega\sum_{l=1}^{f}\sum_{k=1}^{f}\frac{\partial}{\partial\zeta_l}C_h(\frac{\widetilde{\partial}}{\partial z_i},\frac{\partial}{\partial\overline{\zeta_k}})d\zeta_l\wedge d\overline{\zeta_k} = \Lambda_\omega\sum_{l=1}^{f}\sum_{k=1}^{f}\frac{\widetilde{\partial}}{\partial z_i}C_h(\frac{\partial}{\partial\zeta_l},\frac{\partial}{\partial\overline{\zeta_k}})d\zeta_l\wedge d\overline{\zeta_k}\,.$$

The last equality is a obtained using the Bianchi identity. Since Λ_ω and $\frac{\widetilde{\partial}}{\partial z_i}$ commute, the following equality is obtained :

$$\Lambda_\omega\sum_{l=1}^{f}\sum_{k=1}^{f}\frac{\widetilde{\partial}}{\partial z_i}C_h(\frac{\partial}{\partial\zeta_l},\frac{\partial}{\partial\overline{\zeta_k}})d\zeta_l\wedge d\overline{\zeta_k} = \frac{\widetilde{\partial}}{\partial z_i}(\Lambda_\omega\omega) = 0\,.$$

This completes the proof of the lemma. □

We already noted that the given condition $H^1(Y_\xi,\,\mathbb{Q})=0$ and the above lemma together imply the equality (2.9). This completes the proof of the theorem. □

If $H^1(Y_\xi,\,\mathbb{Q})\neq 0$, then the Picard group of Y_ξ has continuous part. If ψ gives a locally trivial holomorphic fibration, then the family of line bundle $L\Big|_{Y_x}$ gives an infinitesimal deformation map $\tau\,:\,T_\xi X\longrightarrow H^1(Y_\xi,\mathcal{O}_{Y_\xi})$. It is easy to check that the harmonic $(0,1)$-form $\sum_{k=1}^{f}C_h(\frac{\widetilde{\partial}}{\partial z_i},\frac{\partial}{\partial\overline{\zeta_k}})d\overline{\zeta_k}$ in Lemma 2.10 is the harmonic representative of the

image of the tangent vector $\frac{\partial}{\partial z_i}$ under the homomorphism τ. (See [ST], [BS] for a similar argument.)

3. Applications of the positivity of direct images

Let E be an ample vector bundle of rank r on a projective manifold X. Take $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r) \in \mathbb{N}^r$ such that $\lambda_i \geq \lambda_j$ if $i \leq j$. Let $\Gamma^\lambda E$ denote the vector bundle associated to E for the weight λ. If, in particular, $\lambda_i = 0$ for $i \geq 2$, then $\Gamma^\lambda E$ is the symmetric power $S^{\lambda_1}(E)$; if $\lambda_i = 0$ for $i \geq k+1$ and $\lambda_i = 1$ for $i \leq k$, then $\Gamma^\lambda E$ is the exterior power $\bigwedge^k(E)$. Let

$$\psi \,:\, M_\lambda(E) \longrightarrow X$$

denote the associated flag bundle over X and $L_\lambda \longrightarrow M_\lambda(E)$ is the corresponding line bundle. If $\lambda_2 = 0$, then $M_\lambda(E) = \mathbb{P}(E)$ and $L_\lambda = \mathcal{O}_{\mathbb{P}(E)}(\lambda_1 + r)$. The direct image $\psi_*(L_\lambda \bigotimes K_{M_\lambda(E)/X})$ coincides with $\Gamma^\lambda E \bigotimes (\bigwedge^r E)^{\otimes l}$, where $l \in [1, r]$ such that $\lambda_l \neq 0$ and $\lambda_{l+1} = 0$; $K_{M_\lambda(E)/X}$ is the relative canonical bundle for the projection ψ. In this special setup Theorem 2.3 reads as follows :

Theorem 3.1. *The vector bundle $\Gamma^\lambda E \bigotimes (\bigwedge^r E)^{\otimes l}$ is Nakano positive. In particular, setting $\lambda = (k, 0, \cdots, 0, 0)$ the vector bundle $S^k(E) \bigotimes \det E$ is Nakano positive.*

Theorem 3.1 combined with a vanishing theorem of Nakano proved in [Na] immediately gives as a corollary the following result of Demailly proved in [De].

Corollary 3.2. *Let E be an ample vector bundle on a projective manifold X of dimension n. Then*

$$H^{n,i}(X,\, \Gamma^\lambda E \otimes (\det E)^{\otimes l}) \,=\, 0$$

for $i \geq 1$ and λ as above.

We note that the special case Corollary 3.2 where $\Gamma^\lambda E = S^k(E)$ was proved by Griffiths [Gr].

Remark 3.3. The curvature terms of the L^2-metric on the vector bundle $\Gamma^\lambda E \otimes (\det E)^{\otimes l}$ are

$$\psi_*(\{\nabla_{\overline{\widetilde{\frac{\partial}{\partial z_j}}}} \nabla_{\widetilde{\frac{\partial}{\partial z_i}}} \widehat{t}_\alpha \,, \widehat{t}_\beta\})$$

The curvature of the dual metric is the negative of the transpose of the initial curvature. Hence, the sesquilinear form on $TX \otimes E^\star$ computed with the dual metric is Nakano negative. As an immediate consequence, for an ample vector bundle E we have

$$H^{p,n}(X,\, \Gamma^\lambda E \otimes (\det E)^{\otimes l}) \,=\, 0$$

if $p \geq 1$ and λ as above.

Let X be a compact complex manifold equipped with a Hermitian structure ω. In [DPS], the notion of a *numerically effective line bundle* on X is defined to be a holomorphic line bundle L satisfying the condition that given any $\epsilon > 0$, there is a Hermitian metric h_ϵ on L such that

$$\Theta_{h_\epsilon} \geq -\epsilon \cdot \omega ,$$

where Θ_{h_ϵ} is the Chern curvature for the Hermitian metric h_ϵ [DPS, Definition 1.2]. The manifold X being compact, this condition, of course, does not depend on ω. A vector bundle E over X is called *numerically effective* if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ over $\mathbb{P}(E)$ is numerically effective [DPS, Definition 1.9].

We have the following proposition as an easy consequence of the proof of Theorem 2.3.

Proposition 3.4. *Let X be a compact complex manifold equipped with a Hermitian form ω. A vector bundle E over X is numerically effective if and only if there is a Hermitian metric $h_{k,\varepsilon}$ on $S^k(E)\det E$, for all $k \geq 1$ and all $\varepsilon > 0$, such that*

$$\sqrt{-1} \cdot C_{h_{k,\varepsilon}}(S^k(E) \otimes \det E) > -\varepsilon \cdot \omega \otimes Id_{S^k(E)\otimes \det E} ,$$

where $C_{h_{k,\varepsilon}}(S^k(E) \otimes \det E)$ denotes the curvature of the metric $h_{k,\varepsilon}$. The inequality is in the sense of Nakano.

Proof. Let E be a numerically effective vector bundle of rank r over the compact complex manifold X. Let Y denote the projective bundle $\mathbb{P}(E)$ over X. The natural projection of Y to X will be denoted by ψ.

The tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$ is numerically effective. For any $\varepsilon > 0$, consider the Hermitian metric h_ε on $\mathcal{O}_{\mathbb{P}(E)}(1)$ as in Definition 1.2 (pp. 299) of [DPS]. This Hermitian metric h_ε induces a Hermitian metric on each $\mathcal{O}_{\mathbb{P}(E)}(k)$, where $k > 1$.

Consequently, we have a Hermitian metric $h_{k,\varepsilon}$ on each $S^k(E) \bigotimes \det E$ obtained as the L^2-metric for the above Hermitian metric on $\mathcal{O}_{\mathbb{P}(E)}(k+r)$ obtained from h_ε. Now, from the proof of Theorem 2.3 it can be deduced that the condition for numerically effectiveness of a line bundle in Definition 1.2 (pp. 299) of [DPS], given in terms of Hermitian metrics h_ε, ensures that the inequality in the proposition is valid.

Conversely, let E be a vector bundle such that the inequality condition in the statement of the proposition is valid. Now it follows immediately from the criterion of numerically effectiveness given in Theorem 1.12 (pp. 306) of [DPS] that such a vector bundle E must be numerically effective. This completes the proof of the proposition.□

Remark 3.5. Let E be a vector bundle of rank r and set $L = \mathcal{O}_{\mathbb{P}(E)}(r+k)$ in (2.6). The expression for the curvature of the L^2-metric on $S^kE \otimes \det E$ shows that if E is

ample, then for each $k \geq 1$, there is a Hermitian metrics on $S^kE \otimes \det E$ such that there exits a positive number $\varepsilon > 0$ with the property that for every positive integer k, the inequality

$$\sqrt{-1} \cdot c(S^kE \otimes \det E) > (k+r)\varepsilon\omega \otimes \mathrm{Id}_{S^kE\otimes \det E}$$

is valid, where ω is any fixed Hermitian form on X; the inequality is in the sense of Nakano.

We observe that the above property is actually a characterization of ampleness. Indeed, if E is a vector bundle which satisfies the above condition, then fix any metric on the line bundle $\det E$. By subtracting it to $S^kE \otimes \det E$, we get a metric on S^kE whose curvature is, for a large k, Nakano positive. Hence, S^kE must be ample. This immediately yields the ampleness of E.

References

[BS] I. Biswas and G. Schumacher : Determinant bundle, Quillen metric, and Petersson-Weil form on moduli spaces. Geom. Funct. Anal. **9** (1999), 226–255.

[De] J.-P. Demailly : Vanishing theorems for tensor powers of an ample vector bundle. Invent. Math. **91** (1988), 203–220.

[DS] J.-P. Demailly and H. Skoda : Relations entre les notions de positivité de P.A. Griffiths et de S. Nakano. Séminaire Pierre Lelong – Henri Skoda (Analyse) Années 1978/79. 304–309, Lecture Notes in Math., 822, Springer, Berlin, 1980.

[DPS] J.-P. Demailly, T. Peternell and M. Schneider : Compact complex manifolds with numerically effective tangent bundles. Jour. Alg. Geom. **3** (1994), 295–345.

[Gr] P.A. Griffiths : Hermitian differential geometry, Chern classes, and positive vector bundles. *Global Analysis*, Papers in Honor of K. Kodaira (Ed. D. C. Spencer and S. Iyanaga), 185–251. Princeton Univ. Press, 1969.

[Ha] R. Hartshorne : Ample vector bundles. Inst. Hautes Études Sci. Publ. Math **29** (1966), 63–94.

[Na] S. Nakano : On complex analytic vector bundles. Jour. Math. Soc. Japan **7** (1955), 1–12.

[Mo] Ch. Mourougane : Images directes de fibrés en droites adjoints. Publ. RIMS, Kyoto Univ. **33** (1997), 893–916.

[ST] G. Schumacher and M. Toma : On the Petersson-Weil metric for the moduli space of Hermite-Einstein bundles and its curvature. Math. Ann. **293** (1992), 101–107.

[Um] H. Umemura : Some results in the theory of vector bundles. Nagoya Math. Jour. **52** (1973), 97–128.

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, INDIA

E-mail address: `indranil@math.tifr.res.in`

Institut de Mathématiques de Jussieu, Boîte 247, 4, place Jussieu, 75 252 Paris Cedex 05, FRANCE

E-mail address: `mourouga@math.jussieu.fr`